

## Lecture 4

# Frequency Domain Analysis and Fourier Transform

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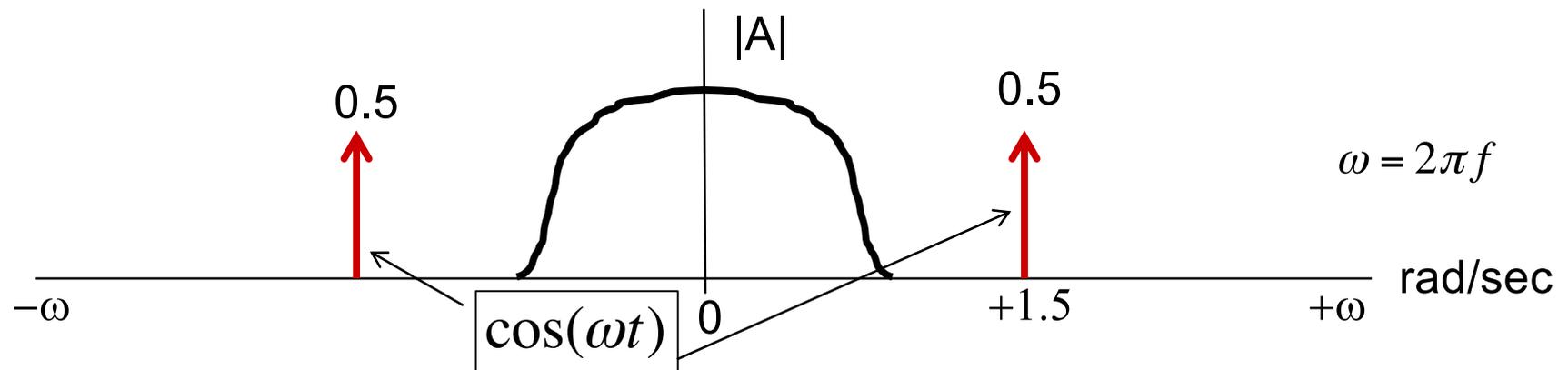
# Relationship between exponentials and sinusoids

- ◆ Euler's formula:

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ e^{-j\omega t} &= \cos(-\omega t) + j\sin(-\omega t) \\ &= \cos(\omega t) - j\sin(\omega t) \end{aligned}$$

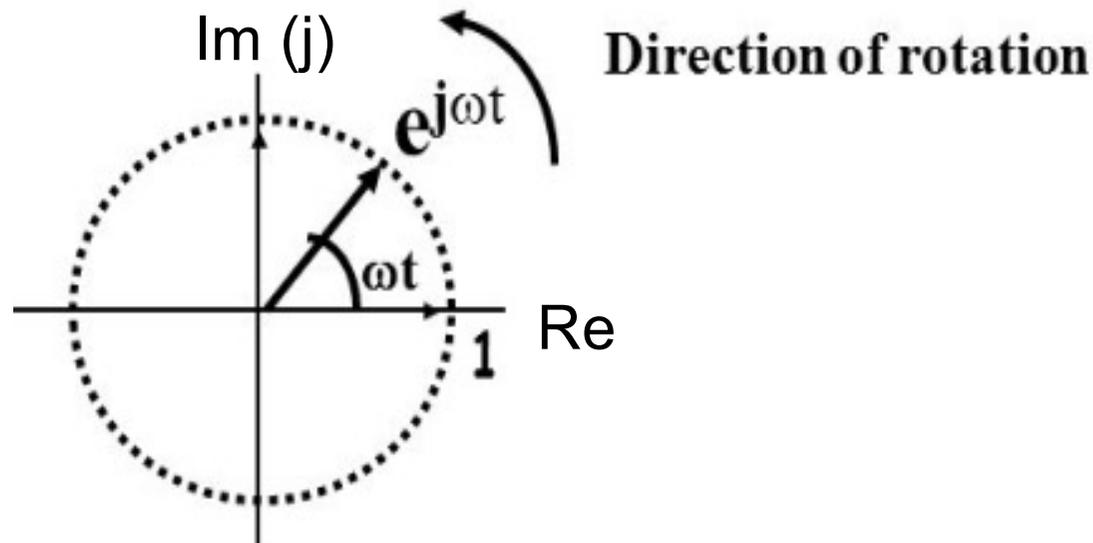
$$\begin{aligned} \cos(\omega t) &= \frac{e^{j\omega t} + e^{-j\omega t}}{2} \\ \sin(\omega t) &= \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \end{aligned}$$

- ◆ Therefore, in signal analysis, we usually regard “frequency” to be  $\omega$  in the exponential vector  $e^{j\omega t}$ .
- ◆ The frequency spectrum is therefore a plot of the amplitude (and phase) projected onto exponential components  $e^{j\omega t}$  for different  $\omega$ .

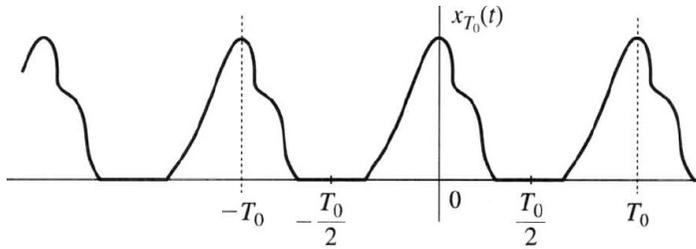


## $e^{j\omega t}$ viewed as a VECTOR

- $e^{j\omega t}$  is the building block, each at different frequency  $\omega$ .
- Can be viewed as a VECTOR as show below.
- The magnitude of the vector  $|e^{j\omega t}|$  is 1.
- This vector is rotating in a complex plane at a rate of  $\omega$  rads/sec in the direction shown.
- $\cos(\omega t)$  and  $\sin(\omega t)$  are just the projection of the this vector on the REAL (x-axis) and IMAGINERY (y-axis) axes in this diagram.



# Fourier Series in three forms



$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t dt \quad b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t dt$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

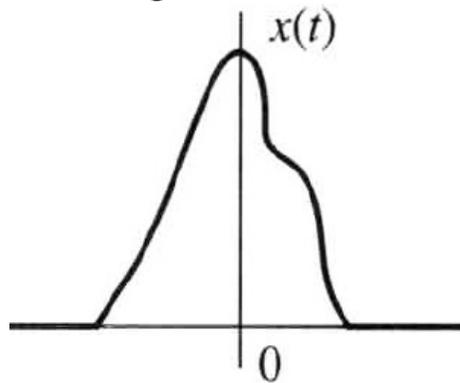
$$C_n = \sqrt{a_n^2 + b_n^2}$$
$$\theta_n = \tan^{-1} \left( \frac{b_n}{a_n} \right)$$

$$x(t) = \sum_{-\infty}^{\infty} D_n e^{j(n\omega_0 t + \theta_n)}$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt$$

# Definition of Fourier Transform

- ◆ The forward and inverse **Fourier Transform** are defined for **aperiodic** signal as:

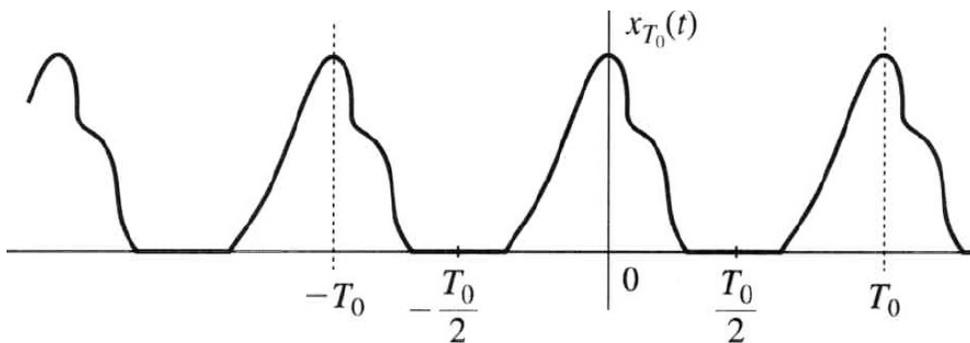


$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$



$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

- ◆ **Fourier series** is used for **periodic** signals.



$$x(t) = \sum_{-\infty}^{\infty} D_n e^{j(n\omega_0 t + \theta_n)}$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t} dt$$

# Define three useful functions

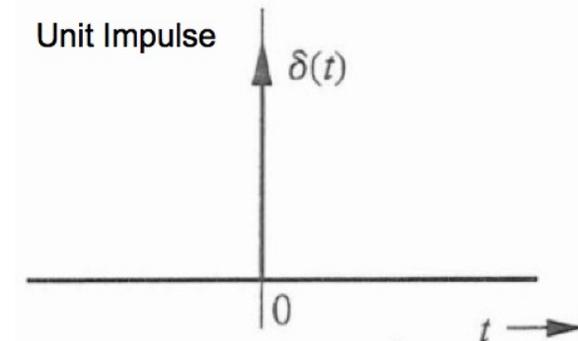
- ◆ A unit rectangular window function **rect(x)**:

$$\text{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

- ◆ The unit impulse function **δ(t)** (Dirac impulse):

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

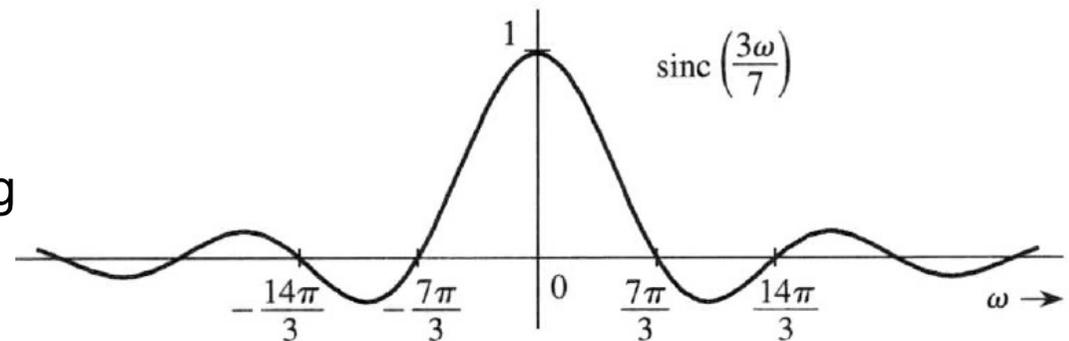
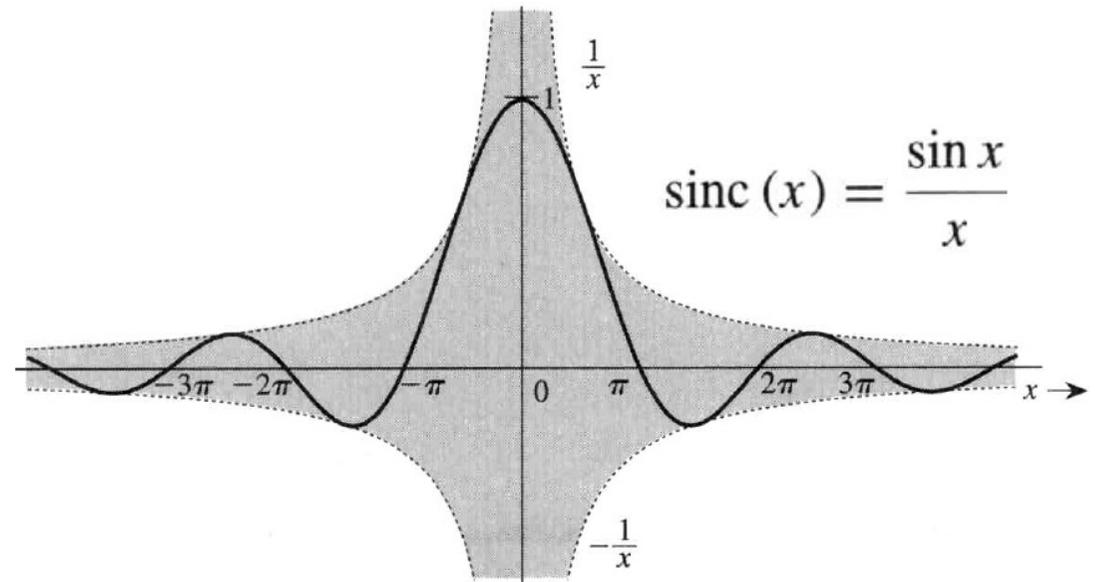


- ◆ Interpolation function **sinc(x)**:

$$\text{sinc}(x) = \frac{\sin x}{x} \quad \text{or} \quad \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

## More about sinc(x) function

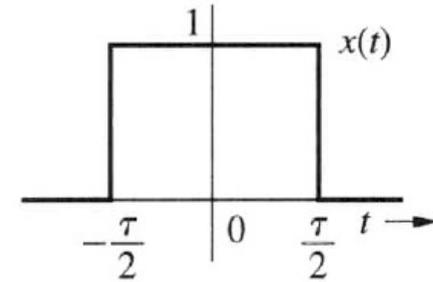
- ◆ **sinc(x)** is an even function of x.
- ◆ **sinc(x)** = 0 when  $\sin(x) = 0$  except when  $x=0$ , i.e.  $x = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$
- ◆ **sinc(0)** = 1 (derived with L'Hôpital's rule)
- ◆ **sinc(x)** is the product of an oscillating signal  $\sin(x)$  and a monotonically decreasing function  $1/x$ . Therefore it is a damping oscillation with period of  $2\pi$  with amplitude decreasing as  $1/x$ .



# Fourier Transform of $x(t) = \text{rect}(t/\tau)$

- ◆ Evaluation:

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

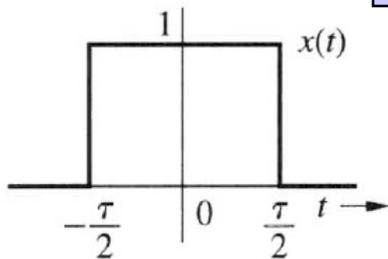


- ◆ Since  $\text{rect}(t/\tau) = 1$  for  $-\tau/2 < t < \tau/2$  and 0 otherwise

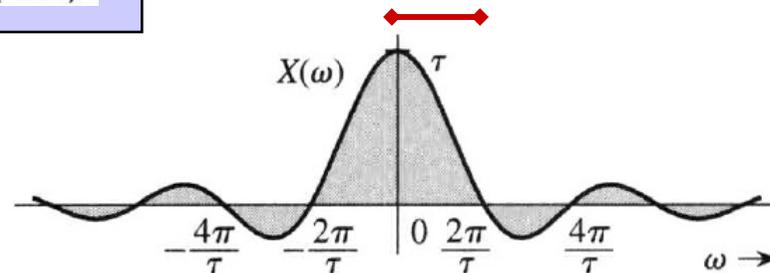
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega} (e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2 \sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\text{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\text{FT}} \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

**Bandwidth  $\approx 2\pi/\tau$**



FT  
↔



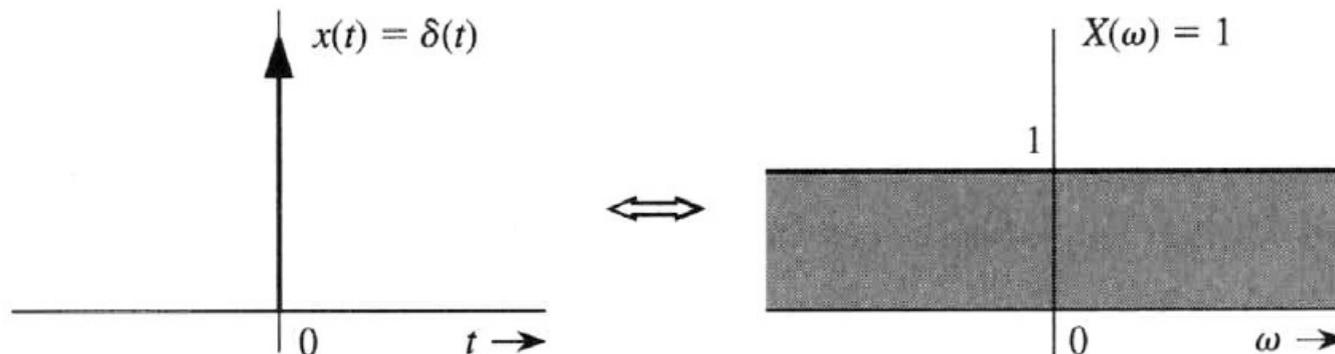
# Fourier Transform of unit impulse $x(t) = \delta(t)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt = 1$$

- IMPORTANT – Unit impulse contains COMPONENT AT EVERY FREQUENCY.

$$\delta(t) \iff 1$$



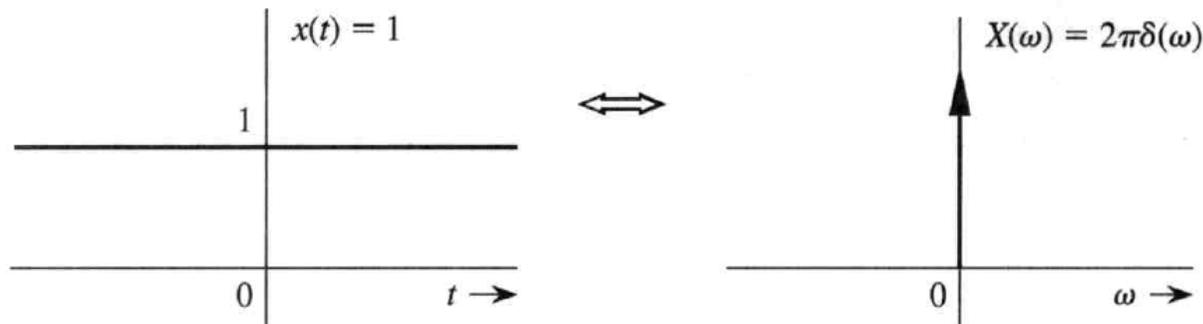
# Inverse Fourier Transform of $\delta(\omega)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

- Spectrum of a constant (i.e. d.c.) signal  $x(t) = 1$  is an impulse  $2\pi\delta(\omega)$ .

$$\frac{1}{2\pi} \iff \delta(\omega) \quad \text{or} \quad 1 \iff 2\pi\delta(\omega)$$



# Inverse Fourier Transform of $\delta(\omega - \omega_0)$

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- ◆ Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

- ◆ Spectrum of an everlasting exponential  $e^{j\omega_0 t}$  is a single impulse at  $\omega = \omega_0$ .

$$\frac{1}{2\pi} e^{j\omega_0 t} \iff \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \iff 2\pi \delta(\omega - \omega_0)$$

and

$$e^{-j\omega_0 t} \iff 2\pi \delta(\omega + \omega_0)$$

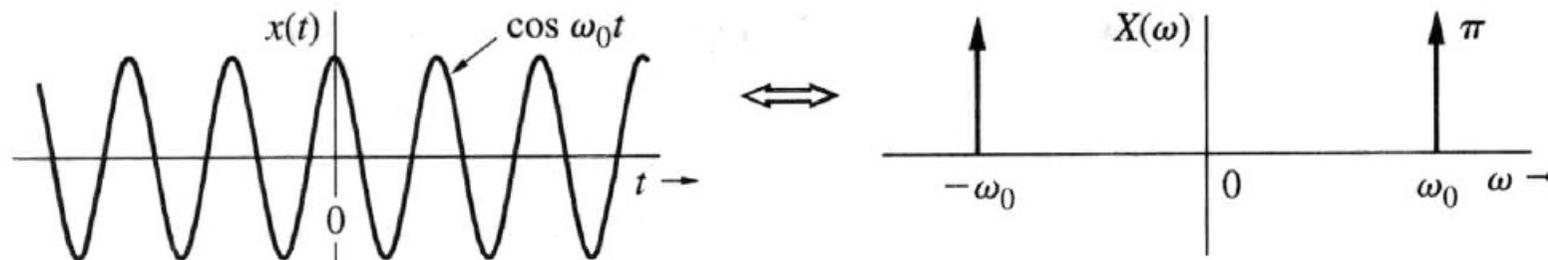
## Fourier Transform of everlasting sinusoid $\cos \omega_0 t$

◆ Remember Euler's formula:  $\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

◆ Use results from previous slide, we get:

$$\cos \omega_0 t \iff \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

◆ Spectrum of cosine signal has two impulses at positive and negative frequencies.



# Fourier Transform of any periodic signal

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- ◆ Fourier series of a periodic signal  $x(t)$  with period  $T_0$  is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \omega_0 = \frac{2\pi}{T_0}$$

- ◆ Take Fourier transform of both sides, we get:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

- ◆ This is rather obvious!

# Fourier Transform of a unit impulse train

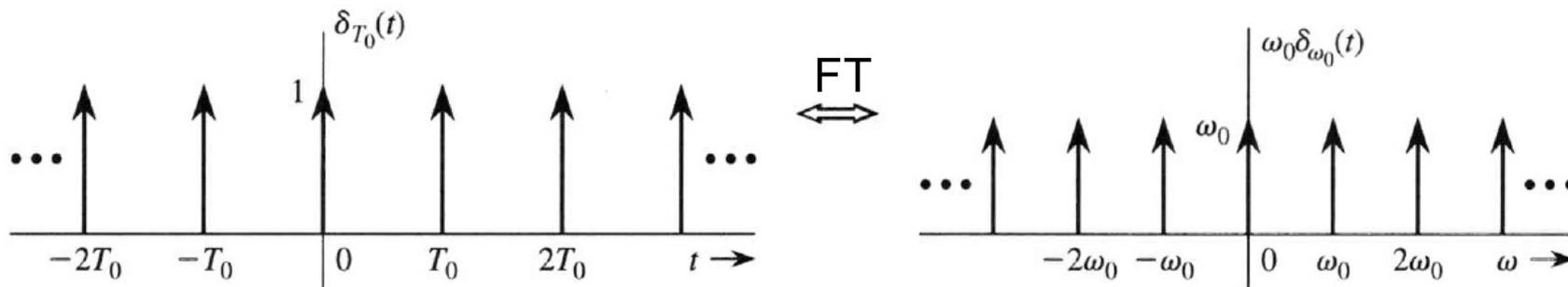
- ◆ Consider an impulse train  $\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - nT_0)$

- ◆ The Fourier series of this impulse train can be shown to be:

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} \quad \text{and} \quad D_n = \frac{1}{T_0}$$

- ◆ Therefore using results from the last slide (slide 13), we get:

$$\begin{aligned} X(\omega) &= \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \quad \omega_0 = \frac{2\pi}{T_0} \\ &= \omega_0 \delta_{\omega_0}(\omega) \end{aligned}$$



# Fourier Transform Table (1)

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No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a - j\omega}$	$a > 0$
3	$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	

## Fourier Transform Table (2)

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No.	$x(t)$	$X(\omega)$	
9	$\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	
12	$\text{sgn } t$	$\frac{2}{j\omega}$	
13	$\cos \omega_0 t u(t)$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$	
14	$\sin \omega_0 t u(t)$	$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$	
15	$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
16	$e^{-at} \cos \omega_0 t u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$

## Fourier Transform Table (3)

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No.	$x(t)$	$X(\omega)$	
16	$e^{-at} \cos \omega_0 t u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$	$a > 0$
17	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi} \text{sinc}(Wt)$	$\text{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi} \text{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$	$\omega_0 = \frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi} e^{-\sigma^2\omega^2/2}$	

# Three Big Ideas

1. **Euler formula** provides an alternative way to represent sine and cosine functions in terms of  $e^{j\omega t}$  and  $e^{-j\omega t}$ .

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

2. **Extracting a portion of a signal**  $x(t)$  for  $-\tau/2 \leq t \leq \tau/2$  can be modelled by multiplying  $x(t)$  by the rectangular function  $\text{rect}(x/\tau)$ .

3. The Fourier Transform of an **infinite train** of unit impulses is again an infinite train of unit impulses.

